Government general degree college, Chapra
Study material for 1st semester general prepared by Aninda Chakraborty

1 Transformation of axes

The coordinates of a point depend on the position of axes. Thus the coordinates of a point and consequently the equation of a locus will be changed with the alteration of origin without the alteration of direction of axes, or by altering the direction of axes and keeping the origin fixed, or by altering the origin and also the direction of axes. Either of these processes is known as transformation of axes or transformation of coordinates.

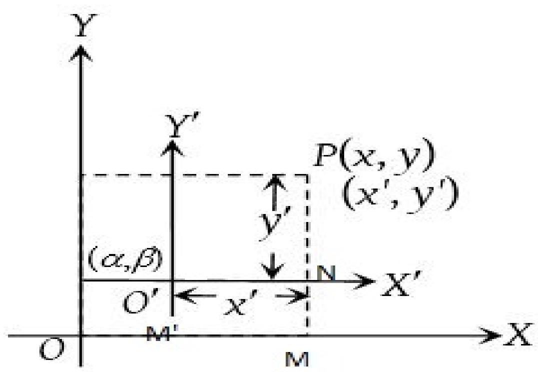

Figure 1:

Let (x, y) be the coordinates of a point P w.r.t rectangular axes OX and OY and (x', y') be the coordinates of it with respect to a new set of axes $O'X'$ and $O'Y'$ which are parallel to the original axes OX and OY respectively (see the Figure-1).
Let (α, β) be the coordinates of the new origin O' w.r.t axes OX and OY. Then from Figure-1 we conclude that,

$$x = OM = OM' + M'M = \alpha + O'N = \alpha + x'$$
$$y = MP = MN + NP = M'O' + y' = \beta + y'$$

&Hence we get,

$$x = x' + \alpha \quad y = y' + \beta$$

Example: Find the form of equation $3x + 4y = 5$ due to change of the origin to the point $(3, -2)$ only.

Solution: Using the above formula we write the transformed equation as $3(x' + 3) + 4(y' - 2) = 5 \implies 3x' + 4y' = 4$

1.1 Rotation of rectangular axes without changing the origin

Let the original axes OX and OY be rotated through an angle θ in counterclockwise direction. Then from the Figure-2 it is clear that,

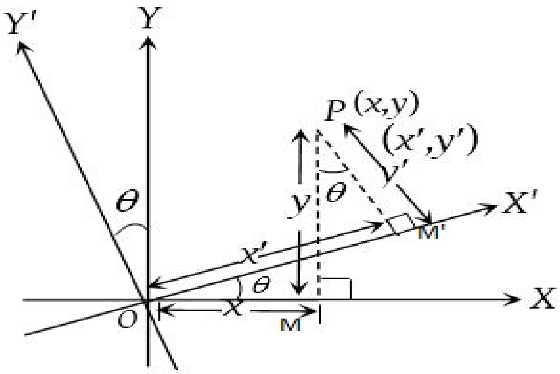

Figure 2:

$$
\begin{aligned}
x &= OM = OPcos(\angle POM) = OPcos(\angle POM' + \angle XOX') \\
&= OPcos(\angle POX')cos(\angle XOX') - OPsin(\angle POX')sin(\angle XOX') \\
&= OM'cos(\angle XOX') - PM'sin(\angle XOX') \\
&= x'cos\theta - y'sin\theta \\
y &= MP = OPsin(\angle POX') = OPsin(\angle POM' + \angle XOX') \\
&= OPcos(\angle POM')sin\theta + OPsin(\angle POM')cos\theta \\
&= OM'sin\theta + M'Pcos\theta = x'sin\theta + y'cos\theta
\end{aligned}
$$

Example: Find the equation of the line $y = \sqrt{3}x$ when the axes are rotated through an angle $\frac{\pi}{3}$.

Solution: Applying the transformation the given equation changes as

$$
\begin{aligned}
x'sin\frac{\pi}{3} + y'cos\frac{\pi}{3} =& \sqrt{3}\left(x'cos\frac{\pi}{3} - y'sin\frac{\pi}{3}\right) \\
Or, \frac{\sqrt{3}}{2}x' + \frac{1}{2}y' =& \sqrt{3}\left(\frac{1}{2}x' - \frac{\sqrt{3}}{2}y'\right) \\
Or, \frac{1}{2}y' + \frac{\sqrt{3}}{2}y' =& 0 \\
Or, y' = 0
\end{aligned}
$$

1.2 Combination of translation and rotation

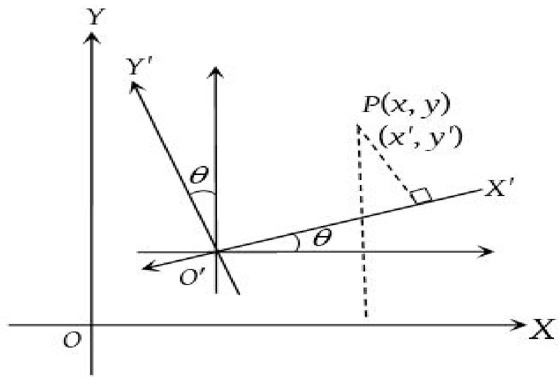

Figure 3:

If the origin O of a set of rectangular axes (OX, OY) is shifted to $O'(\alpha, \beta)$ without changing the direction of axes and then the axes are rotated through an angle θ in the anticlockwise direction (see the Figure-3), the total effective changes in the coordinates (x, y) of a point are given by,

$$x = \alpha + x'' cos\theta - y'' sin\theta$$
$$y = \beta + x'' sin\theta + y'' cos\theta$$

(x'', y'') are the coordinates of the point referred to the final set of axes.

1.3 Invariants under orthogonal transformation

Some expression remain unchanged under an orthogonal transformation. These are known as invariant of orthogonal transformation.

(i) The degree of an equation is an invariant under orthogonal transformation.

(ii) The distance between two points is an invariant under an orthogonal transformation.

(iii) The coefficient of $x^2, xy\ and\ y^2$ and Δ obtained from $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ are invariant under translation.

(iv) $a + b, ab - h^2, f^2 + g^2$ and Δ obtained from $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ are invariant under transformation od rotation.

Here we prove 3rd proposition and remaining are left to the reader. **Solution:** Let the origin be shifted to (α, β). The expression transformed to

$$a(x' + \alpha)^2 + 2h(x' + \alpha)(y' + \beta) + b(y' + \beta)^2 + 2g(x' + \alpha) + 2f(y' + \beta) + c$$
$$ax'^2 + 2hx'y' + by'^2 + 2(a\alpha + h\beta + g)x' + 2f(h\alpha + b\beta + f)y' + a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c$$
$$= a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + c'$$

Where $a' = a, b' = b, h' = h, g' = a\alpha + h\beta + g, f' = h\alpha + b\beta + f, c' = a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c$
We see that the coefficients of x^2, y^2 and xy i.e, a, b and h remain invariant due to translation.

Let us consider the invariance of $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

After translation Δ changes to Δ'

Now $\Delta' = \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix} = \begin{vmatrix} a & h & a\alpha + h\beta + g \\ h & b & h\alpha + b\beta + f \\ a\alpha + h\beta + g & h\alpha + b\beta + f & a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c \end{vmatrix}$

$= \begin{vmatrix} a & h & a\alpha + h\beta + g \\ h & b & h\alpha + b\beta + f \\ a\alpha + h\beta + g & h\alpha + b\beta + f & g\alpha + f\beta + c \end{vmatrix}$ (on subtracting α times the elements of the first column and β times the element of 2nd column from those of the third column)

$= \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$(on subtracting α times the elements of the first row and β times the element of 2nd row from those of the third row)

$\therefore \Delta$ is invariant.

2 General second degree equation

The general equation of 2nd degree in x and y usually written in the form:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{1}$$

The curve represented by this equation is a conic section or simply a conic. The curve is also called a second order curve. The nature of the conic is determined by the quantities

$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, D = ab - h^2$ and $P = a + b$.

In case of rectangular coordinate axes the quantities Δ, D, P are invariants under any orthogonal transformation.

(i) If $\Delta = 0$, the equation (1) represents a pair of straight line.

(ii) If $a = b$ and $h = 0$,the equation represents a circle.

(iii) If $\Delta \neq 0$, the equation represents a proper conic. Here D determine the nature of the conic.

   (a) when $D = 0$, i.e. $ab = h^2$, the conic is a parabola. In this case the 2nd degree terms form a perfect square.
   (b) when $D > 0$,i.e. $ab > h^2$, the conic is a ellipse.
   (c) when $D < 0$,i.e. $ab < h^2$, the conic is a hyperbola. If $a + b = 0$, the conic is a rectangular hyperbola.

2.1 Reduction to canonical form:

The general equation of a 2nd degree can be reduced to the standard equation of a conic by suitable transformation of coordinates. The standard equation is also called the canonical equation or normal canonical form of the equation.

To find the canonical form from the general equation the following transformations are made successively.

(i) The terms in xy is removed by suitable rotation of axes.

(ii) One or both (when possible) the terms in x and y are removed by translation.

(iii) The constant is removed if possible.

Let the axes be rotated through an angle θ. The new coordinates are related as $x = x'cos\theta - y'sin\theta$ and $y = x'sin\theta + y'cos\theta$

Substituting these values of x and y in the equation (1) we have,

$$\begin{aligned} a(x'cos\theta - y'sin\theta)^2 + 2h(x'cos\theta - y'sin\theta)(x'sin\theta + y'cos\theta) + b(x'sin\theta + y'cos\theta)^2 \\ + 2g(x'cos\theta - y'sin\theta) + 2f(x'sin\theta + y'cos\theta) + c = 0 \end{aligned}$$

$$\begin{aligned} Or, (acos^2\theta + 2hsin\theta cos\theta + bsin^2\theta)x'^2 + 2\{h(cos^2\theta - sin^2\theta) - (a-b)sin\theta cos\theta\}x'y' \\ +(asin^2\theta - 2hsin\theta cos\theta + bcos^2\theta)y'^2 + 2(gcos\theta + fsin\theta)x' + 2(fcos\theta - gsin\theta)y' + c = 0 \end{aligned} \quad (2)$$

Let us choose θ in such a way that the coefficient of $x'y'$ in above equation will be zero. To satisfy this condition, we have

$$\begin{aligned} &h(cos^2\theta - sin^2\theta) = (a-b)sin\theta cos\theta \\ &Or, tan(2\theta) = \frac{2h}{a-b}, i.e. \theta = \frac{1}{2}tan^{-1}\frac{2h}{a-b}. \end{aligned}$$

For this value of θ the equation(2) will be of the form: $Ax'^2 + By'^2 + 2Gx' + 2Fy' + C = 0$

Example: Rotate the coordinate axes so that the equation $9x^2 + 24xy + 16y^2 + 2x - 164y + 69 = 0$ transformed into an equation with no xy term.

Solution: We have $A = 9, B = 12, C = 16$, so
$tan(2\theta) = \frac{24}{9-16} = -\frac{24}{7}$
$cos(2\theta) = -\frac{7}{25}, cos(\theta) = \frac{3}{5}, sin(\theta) = -\frac{4}{5}$
Hence the equations of rotation that will accomplish our purpose are

$$\begin{aligned} x &= \frac{3}{5}x' - \frac{4}{5}y', \\ y &= \frac{4}{5}x' + \frac{3}{5}y'. \end{aligned}$$

When we substitute these into the given equation and simplify, we obtain $25x'^2 - 130x' - 100y' + 69 = 0$

3 Pair of straight lines

From the previous discussion we know that the general 2nd degree equation represent a pair of straight lines if $\Delta = 0$ and so we assume $\Delta = 0$. Let the equation of two lines be $lx + my + n = 0$ and $l_1x + m_1y + n_1 = 0$. Then

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (lx + my + n)(l_1x + m_1y + n_1)$$

Comparing the coefficients, we get

$$\begin{aligned} ll_l &= a \qquad lm_l + l_1m = 2h \\ mm_1 &= b \qquad ln_l + l_1n = 2g \\ nn_1 &= b \qquad mn_l + m_1n = 2f \end{aligned}$$

Example: The gradient of one of the lines $ax^2 + 2hxy + by^2 = 0$ is twice that of the other. Show that $8h^2 = 9ab$.

Solution: The equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines passing through the origin. Let the lines be $y = m_1x$ and $y = m_2x$. Then

$$ax^2 + 2hxy + by^2 = (y - m_1x)(y - m_2x)$$

Equating the coefficients of xy and x^2 on both sides, we get

$$\begin{aligned} m_l + m_2 &= \frac{-2h}{b} \\ m_lm_2 &= \frac{a}{b} \end{aligned}$$

Here, it has been given that $m_2 = 2m_1$. From above we get,$3m_l = \dfrac{-2h}{b} \implies m_1 = \dfrac{-2h}{3b}$

Also $2\left(\dfrac{-2h}{3b}\right)^2 = \dfrac{a}{b} \implies 8h^2 = 9ab.$

3.1 point of intersection of two straight lines

Let the general 2nd degree equation $ax^2 + 2hxy + by^2$ represent two intersecting lines and their equations be $lx + my + n = 0$ and $l_1x + m_1y + n_1 = 0$.

Solving the above equations and we get, $x = \dfrac{mn_1 - m_1n}{lm_1 - l_1m}, y = \dfrac{nl_1 - n_1l}{lm_1 - l_1m}$.

So the point of intersection is $\left(\dfrac{mn_1 - m_1n}{lm_1 - l_1m}, \dfrac{nl_1 - n_1l}{lm_1 - l_1m}\right)$

We consider the following example:

Example: Find the value of λ so that the equation $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$

Solution: We have $\lambda x^2 + 10xy + 12y^2 + 5x - 16y - 3 = 0.$
Comparing with the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get,
$a = \lambda, h = -5, b = 12, g = \dfrac{5}{2}, f = -8, c = -3$
Since the given equation represent pair of straight lines and so $\Delta = 0$.

Now here $\Delta = \begin{vmatrix} \lambda & -5 & \frac{5}{2} \\ -5 & 12 & -8 \\ \frac{5}{2} & -8 & -3 \end{vmatrix} = 0 \implies \lambda = 2$

Then $2x^2 - 10xy + 12y^2 + 5x - 16y - 3 = (2x - 4y + l)(x - 3y + m)$
Equating the coefficients of x, y and constant terms,
$2m + l = 5$ and $4m + 3l = 16$
Solving the above equation and we get, $l = 6, m = \dfrac{-1}{2}$
Therefore, the two lines are $x - 2y + 3 = 0$ and $2x - 6y - 1 = 0$.
Solving these two equations, we get the point of intersection as $\left(4, \dfrac{7}{2}\right)$.

3.2 Angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$

Let the lines be $y = m_1x$ and $y = m_2x$. Then from previous discussion we see

$$m_l + m_2 = \frac{-2h}{b}$$
$$m_l m_2 = \frac{a}{b}$$

Let θ be the angle between the lines given by $ax^2 + 2hxy + by^2 = 0$. Then the angle between the lines is given by $tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \frac{\pm\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{m_1 + m_2} = \frac{\pm\sqrt{h^2 - ab}}{a + b}$

$\therefore \theta = tan^{-1}\frac{\pm\sqrt{h^2 - ab}}{a + b}$

The positive sign gives the acute angle between the lines and the negative sign gives the obtuse angle between them.

Corollary 1: If the lines are parallel or coincident, then $\theta = 0$. Then $tan\theta = 0$. Therefore, from above we get $h^2 = ab$.

Corollary 2: If the lines are perpendicular then $\theta = \frac{\pi}{2}$ and so we get from above $tan\left(\frac{\pi}{2}\right) = \frac{\pm\sqrt{h^2 - ab}}{a + b}$. This means $a + b = 0$. Hence, the condition for the lines to be perpendicular is $a + b = 0$ (i.e.) Coefficient of x^2 + Coefficient of $y^2 = 0$.

3.3 Equation for the bisector of the angles between the lines given by $ax^2 + 2hxy + by^2 = 0$

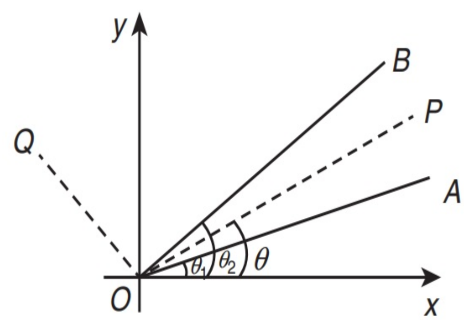

Figure 4:

Let (x, y) be a point on the bisector OP. Then from Figure-4 we get,

$$tan\theta = \frac{y}{x} \implies tan(2\theta) = \frac{2\frac{y}{x}}{1 - \left(\frac{y}{x}\right)^2} = \frac{2xy}{x^2 - y^2}$$

Also $2\theta = \theta_1 + \theta_2 = \frac{tan\theta_1 + tan\theta_2}{1 - tan\theta_1 tan\theta_2} = \frac{\frac{-2h}{b}}{1 - \frac{a}{b}} = \frac{2h}{a - b}$.

So the equation of bisector is $\frac{2xy}{x^2 - y^2} = \frac{2h}{a - b}$

3.4 Equation of two lines joining the origin to the points in which a line meets a conic

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ be the equation of the locus and $lx + my + n = 0$ be the equation of the straight line.

The equation of the line can be written as $\dfrac{lx + my}{-n} = 1$.

Let us make the equation of the locus homogeneous in x and y of 2nd degree by the help of above equation in the following way.

$$ax^2 + 2hxy + by^2 + 2(gx + fy)\frac{lx + my}{-n} + c\left(\frac{lx + my}{-n}\right)^2 = 0 \tag{3}$$

it is of the form $Ax^2 + 2Hxy + By^2 = 0$

The equation (3) obviously represents a pair of straight lines through the origin. Moreover it passes through the points of intersection of the locus and the line. Hence the equation (3) is the required equation.

4 Polar equation of a straight line

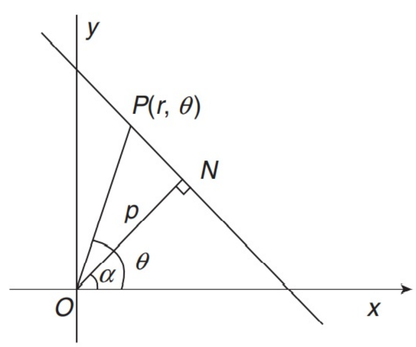

Figure 5:

Let (r, θ) be the coordinates of a point P on the line PN w.r.t the polar O and the initial line OX (see the Figure-5). ON is perpendicular to the line.

Let $ON = p$ and $\angle XON = \alpha$.

Now $ON = OPcos(\theta - \alpha)$, or $p = rcos(\theta - \alpha)$.

It is the polar equation of the line.

The general form of a polar equation of a straight line is $Acos\theta + Bsin\theta - \dfrac{1}{r} = 0$.

If it passes through (r_1, θ_1) and (r_2, θ_2), then

$$Acos\theta_1 + Bsin\theta_1 - \frac{1}{r_1} = 0 \tag{4}$$

$$Acos\theta_2 + Bsin\theta_2 - \frac{1}{r_2} = 0 \tag{5}$$

Eliminating A, B from above equation (4)and main equation we get, $\begin{vmatrix} cos\theta & sin\theta & \frac{1}{r} \\ cos\theta_1 & sin\theta_1 & \frac{1}{r_1} \\ cos\theta_2 & sin\theta_2 & \frac{1}{r_2} \end{vmatrix} = 0$

5 Polar equation of a circle

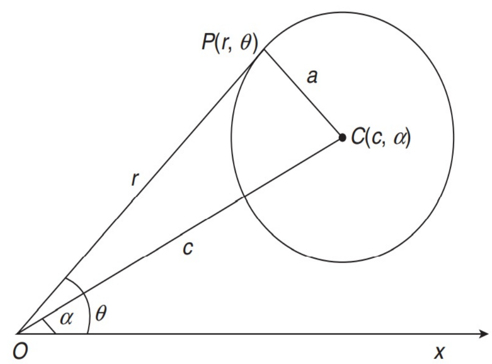

Figure 6:

Let O be the pole and OX be the initial line. Let $C(c, \alpha)$ be the polar coordinates of the centre of the circle. Let $P(r, \theta)$ be any point on the circle. Then $\angle COP = \theta - \alpha$. Let α be the radius of the circle.

$$In, \Delta COP, CP^2 = OC^2 + OP^2 - 2OC.OPcos(\theta - \alpha) \tag{6}$$
$$i.e, a^2 = c^2 + r^2 - 2crcos(\theta - \alpha) \tag{7}$$

This is the polar equation of the required circle.

Corollary 1: If the pole lies on the circumference of the circle then $c = a$. Then the equation of the circle becomes,

$$a^2 = a^2 + r^2 - 2arcos(\theta - \alpha)$$
$$r = 2arcos(\theta - \alpha)$$

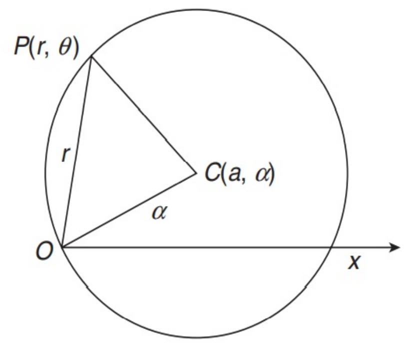

Figure 7:

The equation of the circle $r = 2arcos(\theta - \alpha)$ can be written in the form $r = Acos\theta + Bsin\theta$ in where A and Bare constants (Figure-7).

Corollary 2: If the pole lies on the circumference of the circle and the initial line passes through the centre of the circle then the equation of the circle becomes, $r = 2arcos(\theta)$ since $\alpha = 0$ (Figure-8).

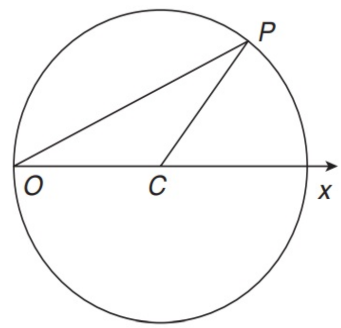

Figure 8:

Corollary 3: Suppose the initial line is a tangent to the circle. Then $c = acosec\alpha$. Therefore, from equation (6) the equation of the circle becomes, $a^2 = a^2cosec^2\alpha + r^2 - 2ar\ cosec\alpha\ cos(\theta - \alpha) + a^2cot^2\alpha$

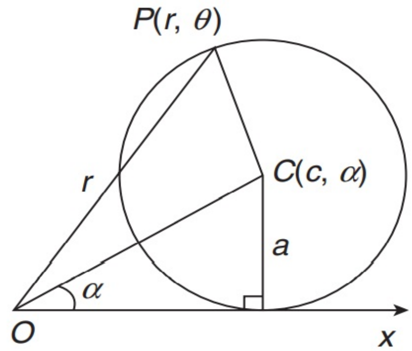

Figure 9:

i.e, $r^2 - 2ar\ cosec\alpha\ cos(\theta - \alpha) + a^2 cot^2\alpha = 0$.

Corollary 4: Suppose the initial line is a tangent and the pole is at the point of contact. In this case $\alpha = 90°$. The equation of the circle becomes,$r = 2asin\theta$.

6 Polar Equation of a Conic

Let S be a fixed point (called the focus) and XM, a fixed straight line (called the directrix) in a plane (Figure-10). Let e be a fixed positive number (called the eccentricity). Then the set of all points P in the plane such that $\dfrac{SP}{PM} = e$ is called a conic section. The conic is

(i) an ellipse if $e < 1$.

(ii) an parabola if $e = 1$.

(iii) an hyperbola if $e > 1$.

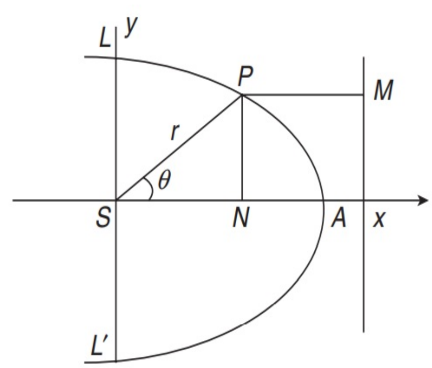

Figure 10:

Derivation of equation:
Let S be focus and XM be the directrix (Figure-10). Draw SX perpendicular to the directrix. Let S be the

pole and SX be the initial line. Let $P(r,\theta)$ be any point on the conic; then $SP = r, \angle XSP = \theta$. Draw PM perpendicular to the directrix and PN perpendicular to the initial line.
Let LSL' be the double ordinate through the focus (latus rectum).
The focus directrix property is

$$\frac{SP}{PM} = e$$
$$\implies SP = ePM = eNX = e(SX - SN)$$
$$\implies r = e\left(\frac{l}{e} - rcos\theta\right) = l - ercos\theta$$
$$Or, \frac{l}{r} = 1 + ecos\theta$$

This is the required polar equation of the conic.

7 Equation of Chord Joining two Points

Here we derive the equation of chord joining the Points whose vectorial angles are $\alpha - \beta$ and $\alpha + \beta$.
Let the equation of the conic be $\frac{l}{r} = 1 + ecos\theta$.
Let the equation of the chord PQ be $\frac{l}{r} = Acos\theta + Bcos(\theta - \alpha)$.

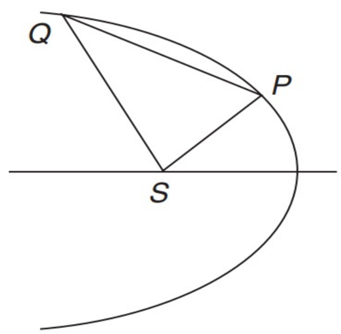

Figure 11:

This chord passes through the point $(SP, \alpha - \beta)$ and $(SQ, \alpha + \beta)$

$$\frac{l}{SP} = Acos(\alpha - \beta) + Bcos(\beta) \tag{8}$$
$$\frac{l}{SQ} = Acos(\alpha + \beta) + Bcos(\beta) \tag{9}$$

Also these two points lie on the conic $\frac{l}{r} = 1 + ecos\theta$,

$$\frac{l}{SP} = 1 + ecos(\alpha - \beta) \tag{10}$$
$$\frac{l}{SQ} = 1 + ecos(\alpha + \beta) \tag{11}$$

From equations (8) and (10) we get,

$$Acos(\alpha-\beta)+Bcos\beta=1+ecos(\alpha-\beta)$$
$$Acos(\alpha+\beta)+Bcos\beta=1+ecos(\alpha+\beta)$$

Subtracting above two equations and we get,

$$A[cos(\alpha-\beta)-cos(\alpha+\beta)]=e[cos(\alpha-\beta)-cos(\alpha+\beta)] \implies A=e$$

Also we get,

$$ecos(\alpha-\beta)+Bcos\beta=1+ecos(\alpha-\beta) \implies B=sec\beta$$

The equation of the chord PQ is $\frac{l}{r}=ecos\theta+sec\beta cos(\theta-\alpha)$

7.1 Tangent at the Point whose Vectorial Angle is α on the conic $\frac{l}{r}=1+ecos\theta$

The equation of the chord joining the points with vectorial angles $\alpha-\beta$ and $\alpha+\beta$ is $\frac{l}{r}=ecos\theta+sec\beta cos(\theta-\alpha)$
This chord becomes the tangent at α if $\beta=0$.
The equation of tangent at α is $\frac{l}{r}=ecos\theta+cos(\theta-\alpha)$.

7.2 Equation of Normal at the Point whose Vectorial Angle is α on the conic

The equation of the conic is $\frac{l}{r}=1+ecos\theta$.
The equation of tangent at α on the conic $\frac{l}{r}=1+ecos\theta$ is $\frac{l}{r}=ecos\theta+cos(\theta-\alpha)$.
The equation of the line perpendicular to this tangent is $\frac{k}{r}=ecos\left(\theta+\frac{\pi}{2}\right)+cos\left(\theta+\frac{\pi}{2}-\alpha\right)$.
i.e, $\frac{k}{r}=-esin\theta-sin(\theta-\alpha)$
If this perpendicular line is normal at P, then it passes through the point (SP,α).
$\frac{k}{SP}=-esin\alpha$ or, $k=-SP.esin\alpha$
Since the point (SP,α) also lies on the conic $\frac{l}{r}=1+ecos\theta$, we have
$\frac{l}{SP}=1+ecos\alpha \implies SP=\frac{l}{1+ecos\alpha}$.
Hence we get, $k=\frac{-lesin\alpha}{1+ecos\alpha}$
Hence, the equation of the normal at α is $\frac{1}{r}\frac{lesin\alpha}{1+ecos\alpha}=esin\theta+sin(\theta-\alpha)$

Some exercise:

1. A chord PQ of a conic subtends a constant 2γ at the focus S and tangents at P and Q meet in T. Prove that $\frac{1}{SP}+\frac{1}{SQ}-\frac{2cos\gamma}{ST}=\frac{2sin^2\gamma}{l}$.
2. A focal chord SP of an ellipse is inclined at an angle α to the major axis. Prove that the perpendicular from the focus on the tangent at P makes with the axis an angle $tan^{-1}\left(\frac{sin\alpha}{e+cos\alpha}\right)$.
3. Show that $r=Acos\theta+Bsin\theta$ represents a circle and find the polar coordinates of the centre.

4. Show that the pair of straight lines joining the origin to the point of intersection of the straight lines $y = mx + c$ and the circle $x^2 + y^2 = a^2$ are at right angles if $2c^2 = a^2(1 + m^2)$.

5. If one of the bisectors of $ax^2 + 2hxy + by^2 = 0$ passes through the point of intersection of the lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ then show that $h(f^2 - g^2) + (a - b)fg = 0$.

6. If the curve $x^2 + y^2 + 2gx + 2fy + c = 0$ intercepts on the line $lx + my = 1$, which subtends a right angle at the origin then show that $a(l^2 + m^2) + 2(gl + fm + 1) = 0$.

7. Show that the radius of a circle remain unchanged due to rigid motion.

8. Show that the distance between the parallel lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $2\sqrt{\dfrac{g^2 - ac}{a(a + b)}}$.

9. Discuss the nature of the conic represented by $11x^2 - 4xy + 14y^2 - 58x - 44y + 71 = 0$.

10. Find the canonical form of the following conic:

    (i) $x^2 + 4xy + y^2 + 4x + y - 15 = 0$

    (ii) $3x^2 + 10xy + 3y^2 - 2x - 14y - 5 = 0$

    (iii) $5x^2 - 20xy - 5y^2 - 16x + 8y - 7 = 0$

    (iv) $x^2 - xy + y^2 - 6x = 0$

    (v) $8x^2 - 12xy + 17y^2 + 16x - 12y + 3 = 0$

11. Show that the equation $(a^2 + 1)x^2 + 2(a + b)xy + (b^2 + 1) = c$ represent an ellipse whose area is $\dfrac{\pi c}{ab - 1}$.

References

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